1 Compliance and crack propagation

The strain energy release rate in a loaded cracked body of thickness B is defined as:

$$G = \frac{1}{B}\frac{d}{da}\left(W - U_{el}\right)$$

Where W is the work done on the specimen by an external load, a the crack length and U_{el} is:

$$U_{el} = \frac{P\Delta}{2} = \frac{CP^2}{2} = \frac{\Delta^2}{2C}$$

Where P is the load, Δ is the displacement, and $C = \frac{\Delta}{P}$ is the compliance of the specimen. Let's assume the crack advances by an infinitesimal amount da.

Question 1

Show that for fixed grips (Δ constant):

$$dU_{el} = -\frac{P^2}{2}\left(\frac{dC}{da}\right)da$$

Question 2

Show that for fixed load (P constant):

$$dU_{el} = \frac{P^2}{2}\left(\frac{dC}{da}\right)da$$

Question 3

What is the work done by external load when the crack extends by da under fixed load conditions ? What is it for fixed grips conditions ?

Question 4

Show that for both cases:

$$G = \frac{P^2}{2B}\left(\frac{dC}{da}\right)$$

Question 5

What is the expression for G in a test carried out at constant displacement rate?

2 Energy release rate of bones

The toughness of human and bovine bones was assessed by Norman et al. using compliance calibration curves (J. Biomechanics 28, 1995). Compact tension specimens (Figure 1) with different thicknesses were machined in tibiae and the critical load and compliance were measured as a function of the initial crack length a (Figure 2). Assume bone is an isotropic material with Young's modulus $E = 11.3$ GPa and Poisson's ratio $\nu = 0.42$.

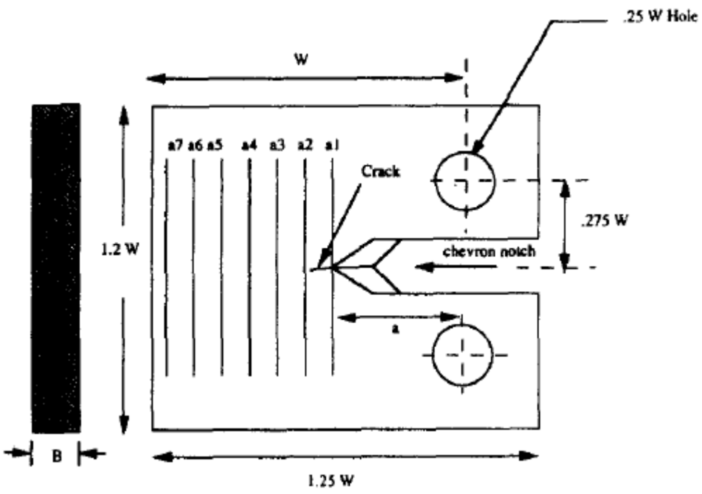

Figure 1: Compact tension specimen

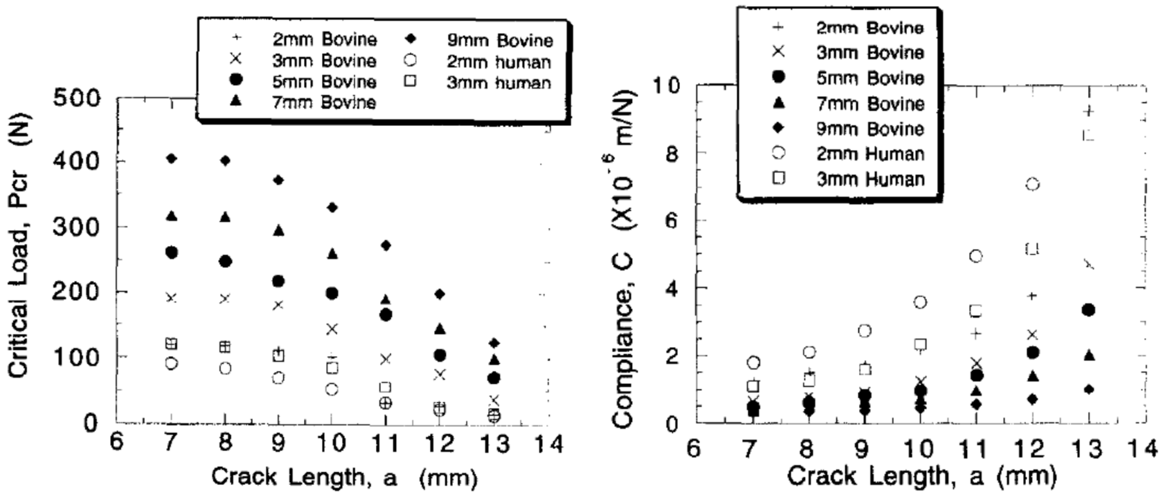

Figure 2: Critical load and compliance of bone specimens as a function of the crack length

Question 1

Estimate $R = G_c$ for the 2mm thick human bone sample, using critical load data for 8mm $< a <$ 11mm (approximate the derivates by finite differences).

Question 2

Does the thicker bone sample have a higher or lower G_c than the 2mm thick sample ?

3 Griffith energy balance of a double cantilever beam

The double cantilever beam (DCB) set-up, also unofficially known as the chopsticks problem, is a classical Mode-I fracture test for materials. As presented in Figure 3, a thin layer of elastic material (of modulus E) is cleaved by a progressive spreading of its left ends. The geometry is assumed to satisfy Bernoulli beam theory and the specimen is characterized by its bending inertia I.

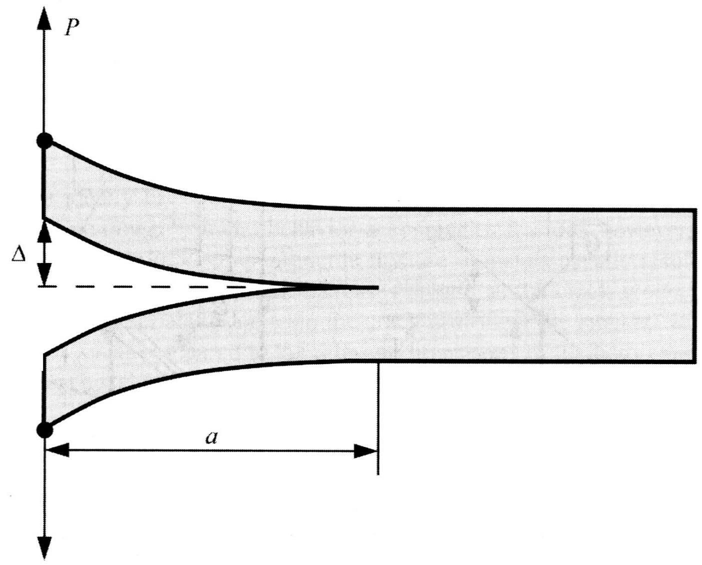

Figure 3: Double cantilever beam set-up

Question 1

Griffith energy balance will be used to describe this problem. Using your structural mechanics knowledge, express the elastic strain energy stored in a Bernoulli beam of length a submitted to a bending moment $M(x)$ along its axis:

$$E_{el} = \int_0^a \Pi(M(x))dx, \tag{1}$$

Π being a function of the bending moment to determine. Reminder:

$$I = \int_A y^2 dA \tag{2}$$

with A the cross section area and y the transverse direction.

Question 2

If we assume that the beams are perfectly clamped at $x = a$, the opening displacement Δ can be related to the force P by

$$\Delta = \frac{Pa^3}{3EI}. \tag{3}$$

Using linear elastic fracture mechanics (LEFM) assumptions, justify the choice of this kinematic and emphasize its limitations.

Question 3

Griffith's remarkable idea was to view fracture as a thermodynamic problem such that, at equilibrium, the free (or the total) energy U of the system is minimized. In our problem, we only consider three energetic contributions:

$$U(a) = E_{el} - W_{ext} + E_{frac}. \tag{4}$$

W_{ext} is the work of external forces and E_{frac} is the fracture energy required to create two new surfaces (surface energy $\equiv \gamma$. Express the total energy of the system for a crack of length a.

Question 4

We want to study now the equilibrium of the system. Two situations are then possible; either Δ has a fixed-value (displacement-control set-up) or P is constant (load-control set-up). For a displacement-control set-up, compute

$$\left.\frac{dU(a)}{da}\right|_{\Delta} \tag{5}$$

and find the equilibrium crack length a_{equ} minimizing the energy of the system.

Question 5

For a load-control set-up, compute

$$\left.\frac{dU(a)}{da}\right|_{P} \tag{6}$$

and discuss the equilibrium of the system.